Deriving Planck's Solution to the Ultraviolet Catastrophe

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Black body radiation refers to the electromagnetic radiation emitted by an idealized object that absorbs and emits all incident radiation perfectly, with its spectral distribution dependent only on temperature. Think of the black body object as being composed of a large number (on the order of Avogadro's number) of harmonic oscillators, and all radiation that is emitted from the black body is due to vibrations of these harmonic oscillators (e.g. none of the light emitted is reflected from another source). Classical physics, using the Rayleigh-Jeans law, predicted an infinite energy output at short wavelengths in the ultraviolet range, a physical impossibility known as the ultraviolet catastrophe (Fig. 1).

This stark disconnect between experimental data and what was the best theory describing radiation at the time was eventually resolved by Max Planck in 1900 by introducing the idea that electromagnetic energy is quantized, meaning it can only be emitted or absorbed in discrete packets of energy $h\nu$, leading to Planck's law of black body radiation. This groundbreaking assumption laid the foundation for quantum mechanics, inspiring later developments such as Einstein's photoelectric effect and Bohr's atomic model. Here, we will derive Planck's equation for the spectrum produced by black body radiation, beginning with his Nobel Prize winning discovery: the quantization of energy.

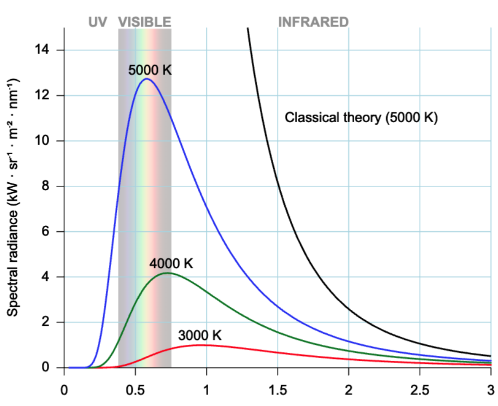

Figure 1: Plot of black body radiation spectra as spectral radiance for different temperatures predicted by Planck's solution, which agrees with experiment (colored lines), compared to spectral radiance as a function of wavelength as predicted by the Rayleigh-Jeans law (black line) [2].

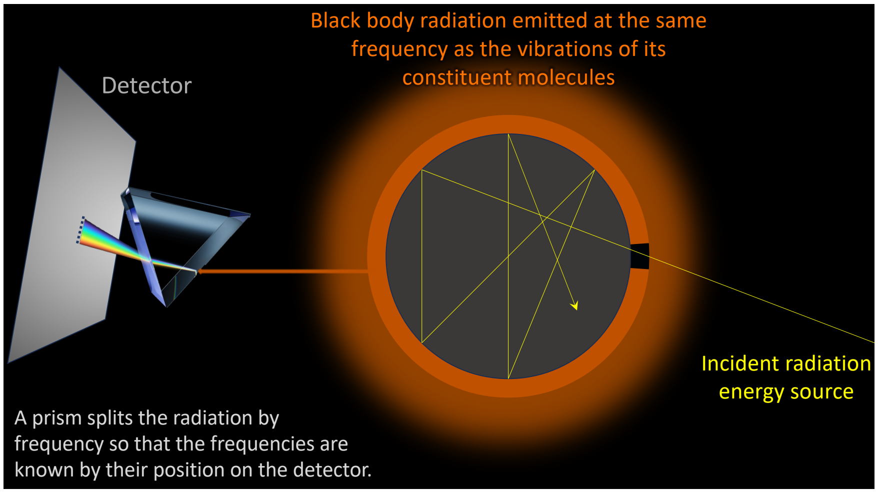

Figure 2: Cartoon representation of a black body radiation experiment.

If we assume that energy is quantized, meaning it cannot take on a continuous range of values but must instead change in discrete steps, then it is reasonable to propose that there exists a fundamental minimum unit of energy. This implies that any allowed energy value must be an integer multiple of this fundamental unit, leading to the expression $nh\nu$, where $h\nu$ represents the smallest indivisible quantum of energy, and n is a non-negative integer. While the frequency ν itself can vary continuously from one physical system to the next, for a given system, the energy exchange at a given frequency still occurs in discrete packets of size $h\nu$, ensuring that the quantization principle remains intact.

To put it simply all matter jiggles, and the hotter something is the more its jiggling. Further, let's consider our black body object to be made up of a collection of harmonic oscillators. If you've seen a weight bouncing on a spring, you know what a harmonic oscillator is! Now because energy is quantized, there is a minimum amount that a jiggling piece of matter can increment upwards in energy, and we can calculate this by multiplying Planck's constant, $h = 6.62607015 \times 10^{-34}\, J/Hz$ (where $Hz = s^{-1}$). Thus, any energy our jiggling system can have will be some integer multiple of $h\nu$, and we can write the following expression for an allowed energetic state for a quantum harmonic oscillator:

$$E_n = nh\nu$$

where n is any non-negative integer. Further, the total energy of a black body object composed of my harmonic oscillators can be calculated by summing over all oscillators at their respective quantum energy state indicated by their quantum number n.

$$E_{Total} = \sum_{n \in N} nh\nu$$

where N is the set of all quantum states for all harmonic oscillators. It is not typically possible to know all quantum states for all harmonic oscillators in a system, but fear not, later we will invoke Boltzmann statistics to calculate the average energy of oscillators, which can be multiplied by the number of oscillators to calcuate E_{total}

Recall that the value measured in the black body radiation experiments is power density as a function of temperature. Power is energy per time and power density is power per volume. Thus, we need to find a way to build up our energy expression $E_n = nh\nu$ into an expression of power per volume. Let us first figure out what the average energy of our quantum harmonic oscillator is at a particular temperature. Fortunately for Max Planck, Ludwig Boltzmann had already introduced the Boltzmann distribution, which can be used to calculate average energy $\langle E \rangle$. First, we write the Boltzmann probability for observing a particular energy E:

$$P(E) = \frac{e^{-E/kT}}{\sum_{n=0}^{\infty} e^{-E_n/kT}}$$

where the sum in the denominator is a normalizing constant we call the partition function.

Since we have expressions for the probabilities and the values of all possible energies, we can use this to calculate a mean. It's the same process by which you could calculate the average age, $\langle age \rangle$, of a group of 10 students if you knew that 3 students are 20, 6 students are 21, and 1 student is 22:

$$\langle age \rangle = \frac{3}{10} \cdot 20 + \frac{6}{10} \cdot 21 + \frac{1}{10} \cdot 22 = 20.8$$

or equivalently, we could think of those fractions as probabilities of randomly selecting a student of each age ($P(age)$):

$$\langle age \rangle = P(20) \cdot 20 + P(21) \cdot 21 + P(22) \cdot 22 = 20.8$$

Just as we sum over all the ages multiplied by the probability of a student being that age, we can do the same with the energies by summing over each energy multiplied by the probability of the oscillator having that energy.

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n \, e^{-E_n/kT}}{\sum_{n=0}^{\infty} e^{-E_n/kT}}$$

Note that the partition function in the denominator is the same for each individual probability, so it factors out, allowing us to divide the entire sum in the numerator by the partition function once. Further, since we have our quantized expression for energy, $E_n = nh\nu$, we can write:

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu \, e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}}$$

Next, we recognize that the infinite sum in the denominator is a geometric series:

$$\sum_{n=0}^{\infty} e^{-nh\nu/kT} = 1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + \dots$$

The geometric series is easier to see if we define x like this:

$$x = e^{-h\nu/kT}$$

Now, rewriting the series:

$$\sum_{n=0}^{\infty} e^{-nh\nu/kT} = \sum_{n=0}^{\infty} x^n$$

Since this is the infinite geometric series that sums to:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1$$

Thus, we substitute back in for x, and the sum in the denominator becomes:

$$\sum_{n=0}^{\infty} e^{-nh\nu/kT} = \frac{1}{1 - e^{-h\nu/kT}}$$

for $|e^{-h\nu/kT}| < 1$, which is fine because negative exponentials are always less than 1, and all the parameters in our negative exponential are non-negative. So we now have a finite value for our partition function, rather than an infinite series!

Next, we look at the sum in the numerator, and see that it is related to that same geometric series by its derivative:

$$\sum_{n=0}^{\infty} nh\nu \; e^{-nh\nu/kT}$$

We define x the same way as before:

$$x = e^{-h\nu/kT}$$

So the sum now simplifies to:

$$h\nu \sum_{n=0}^{\infty} nx^n$$

Note that the constant $h\nu$ has been factored out of the sum. Returning to the geometric series, we can turn it into our new series by applying the differential operator $\frac{d}{dx}$ to it:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

Differentiating both sides with respect to x:

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Multiplying by x on both sides:

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Now we substitute $x = e^{-h\nu/kT}$ back into the expression:

$$\sum_{n=0}^{\infty} ne^{-nh\nu/kT} = \frac{e^{-h\nu/kT}}{(1-e^{-h\nu/kT})^2}$$

Multiplying by $h\nu$, we get:

$$\sum_{n=0}^{\infty} nh\nu\, e^{-nh\nu/kT} = h\nu\frac{e^{-h\nu/kT}}{(1-e^{-h\nu/kT})^2}$$

Now that we have replaced both infinite sums with finite expressions, we can combine our numerator and denominator to replace both sums over n with functions of only ν and T:

$$\langle E\rangle = h\nu\frac{e^{-h\nu/kT}}{(1-e^{-h\nu/kT})^2} \cdot \frac{1-e^{-h\nu/kT}}{1}$$

Simplifying this:

$$\langle E\rangle = h\nu\frac{e^{-h\nu/kT}}{1-e^{-h\nu/kT}}$$

Then by dividing numerator and denominator by $e^{-h\nu/kT}$, we can write:

$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1}$$

Next, recall that we need an expression of energy density for a hot object made up of harmonic oscillators, not just the energy of a single oscillator (later we'll go from energy density to power density). In a hot, vibrating material, energy is distributed across many possible vibrational modes. Since matter itself is discretized, or quantized, because matter is made up of atoms, there is a finite number of vibrational states possible in any object. The density of states tells us how many allowed vibrational states exist per unit volume at a given frequency ν. Although we won't do a full derivation for the density of states expression, $\frac{8\pi\nu^2}{c^3}$, as it requires a deeper discussion of solid state physics, I'll create additional material deriving that later. For now, it will suffice to know that each of these states carries an average energy $\langle E \rangle$, and the total energy density $u(\nu, T)$ is obtained by multiplying the density of states (vibrational modes per unit volume) by the average energy per state.

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}\langle E \rangle$$

Note that the units for the density of states is m^{-3}/Hz, or $m^{-3}s$, so multiplying by energy gives us units of energy per volume per unit frequency. Thus, we multiply the density of states by the average energy:

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT} - 1}$$

Producing our energy density expression as a function of frequency and temperature:

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

It's not quite a power density yet, but were getting closer! Our expression for energy per volume per unit frequency ($u(\nu, T)$) tells us about energy being emitted from the black body, but it does not tell us about the energy being detected by our detector. For this, we need to calculate the spectral energy flux I_ν (also called specific intensity), which is in units of power per area per frequency per steradian (symbol: sr, or square radian, the unit of solid angle - Fig. 4). A flux is a flow, so spectral energy flux I_ν is the amount of energy at frequency ν flowing through the surface of our detector (hence why it is per unit area). The first thing to consider is that the energy propagates as light, so we know how fast the energy will flow through the area of the detector:

$$Energy\ flux = Energy\ density \times Speed$$

If the radiation was all going the same direction, we could measure the total spectral energy flux through our detector by just multiplying energy density by the speed of light:

$$I_\nu = c\,u(\nu, T)$$

In reality, consider that on the surface of this black body, radiation is just as likely to shine outwards as into the cavity, which means that only half of the light will radiate outwards. Further, only the light on the side of a spherical black body facing the detector has the possibility of sending light towards the

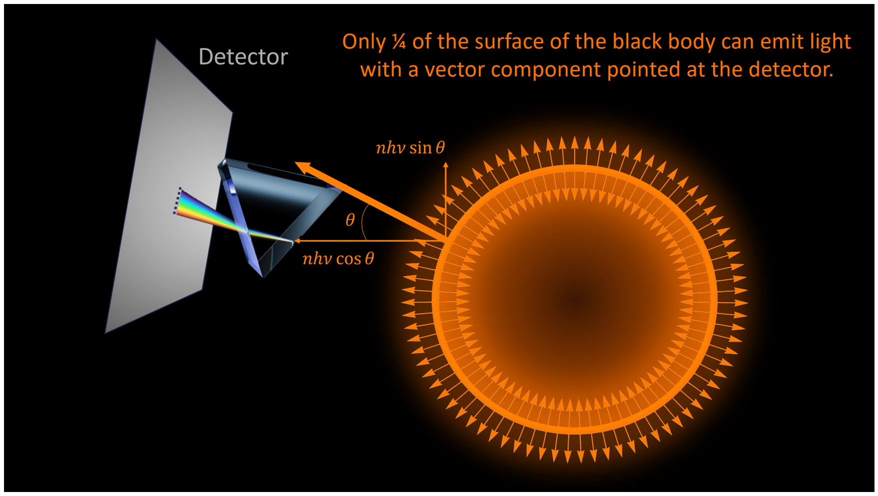

Figure 3: Half of the photons are pointed inwards, and only half of the photons pointed outwards have an x component pointed towards the detector.

detector. These two effects result in an attenuation of the measured energy flux by $1/2 \times 1/2 = 1/4$ (Fig. 3). Thus, we write:

$$I_\nu = \frac{c}{4} u(\nu, T)$$

Next, if we consider that, of the light that does hit the detector, only the component of that field vector that is pointed in the direction of the detector is measured (think of a glancing blow vs. a direct hit), we account for this attenuation by integrating spectral radiance $B_\nu(T)$ times $cos(\theta)$, the vector component of the incident radiation that is directly hitting the detector over all possible angles (Fig. 3), giving us spectral radiance $B_\nu(T)$ is as the power per unit area per unit solid angle per unit frequency, so we relate it to I_ν:

$$I_\nu = \int_{\text{hemisphere}} B_\nu(T) \cos\theta \, d\Omega.$$

Note the integration element $d\Omega = \sin\theta \, d\theta \, d\phi$ is the solid angle element, which you can visualize by thinking about how many vector arrows are packed into a given wedge-shaped segment, as shown in Fig. 4. Hence, we substitute $d\Omega$ into our previous equation:

$$I_\nu = B_\nu(T) \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta.$$

The limits of integration can be justified by looking at Fig. 5, where we note that for an emitter on the surface of a black body object pointed towards the detector, we must also account for the fact that the radiation can emit at any

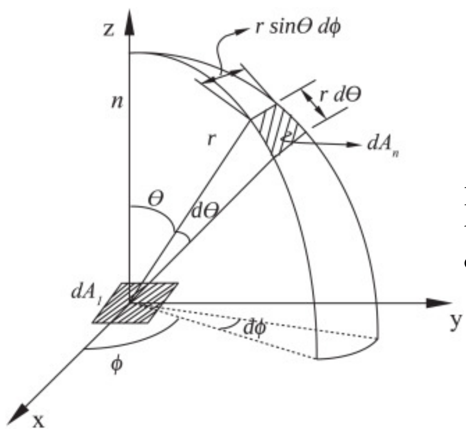

Figure 4: Graphical representation of the wedge-like solid angle element of integration[1].

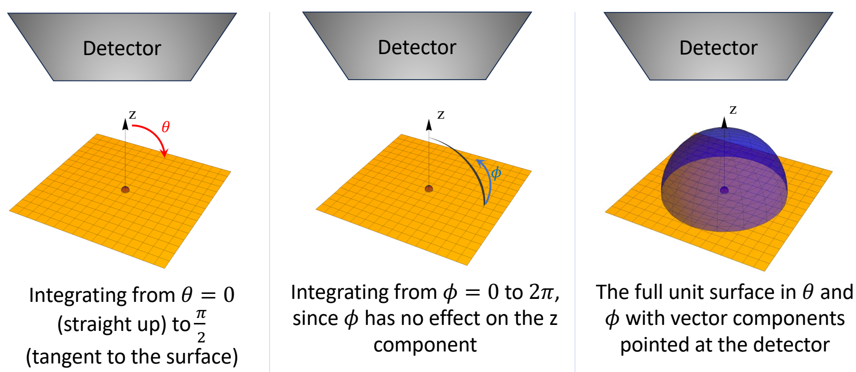

Figure 5: Justification for integration limits over θ and ϕ for a radiation emitter on the surface of a black body object pointed towards the detector.

angle relative to that surface, and only the vector component of the photon emitted aligned with the detector will be measured (Fig. 3).

The previous expression for I_ν then integrates to:

$$I_\nu = \pi B_\nu(T)$$

Now that we have two definitions for I_ν, we can set the two equal to each other:

$$\pi B_\nu(T) = \frac{c}{4} u(\nu, T)$$

Finally, we have an equation for power density, the quantity measured by the experimentalists studying black body radiation!:

$$B_\nu(T) = \frac{c}{4\pi} u(\nu, T)$$

Next, let us expand this expression to include our previously derived expression for energy density $u(\nu, T)$:

$$B_\nu(T) = \frac{c}{4\pi} \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

which simplifies to:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

This is a perfectly fine expression for power density in the frequency domain, but the experimental data was collected in terms of wavelength, not frequency, so we only need to map this expression to a wavelength description, and we're done! In order to express $B_\nu(T)$ in terms of wavelength λ instead of frequency ν. We use the well known relationship:

$$\nu = \frac{c}{\lambda}$$

Differentiating both sides:

$$d\nu = -\frac{c}{\lambda^2} d\lambda$$

Ignoring the negative sign (negative frequencies and negative lengths don't have any physical meaning anyway), we can write:

$$d\nu = \frac{c}{\lambda^2} d\lambda$$

Since spectral radiance must remain the same when changing variables:

$$B_\lambda \, d\lambda = B_\nu \, d\nu$$

Dividing both sides by $d\lambda$:

$$B_\lambda = B_\nu \frac{d\nu}{d\lambda}$$

Substituting $\frac{d\nu}{d\lambda} = \frac{c}{\lambda^2}$ from a couple steps ago:

$$B_\lambda = B_\nu \frac{c}{\lambda^2}$$

Substituting power density in the frequency domain $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$, and replacing $\nu = c/\lambda$:

$$B_\lambda = \left(\frac{2h(c/\lambda)^3}{c^2}\frac{1}{e^{h(c/\lambda)/kT}-1}\right)\frac{c}{\lambda^2}$$

$$B_\lambda = \left(\frac{2hc^3}{c^2\lambda^3}\frac{1}{e^{hc/\lambda kT}-1}\right)\frac{c}{\lambda^2}$$

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

Finally, we have Planck's law in wavelength form:

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

References

[1] https://www.sciencedirect.com/topics/engineering/solid-angle

[2] https://en.wikipedia.org/wiki/Ultraviolet_catastrophe Author: Darth Kule